

A NOTE ON SPECTRAL PROPERTIES OF THE p -ADIC TREE

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ABSTRACT. We study the spectrum of the operator D^*D , where the operator D , introduced in [6], is a forward derivative on the p -adic tree, a weighted rooted tree associated to \mathbb{Z}_p via Michon's correspondence. We show that the spectrum is closely related to the roots of a certain q -hypergeometric function and discuss the analytic continuation of the zeta function associated with D^*D .

1. INTRODUCTION

This note builds on our previous paper [6] which described a new spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for the C^* -algebra of continuous functions on the space of p -adic integers \mathbb{Z}_p . The construction of this spectral triple utilized a coarse-grained approximation of the space \mathbb{Z}_p and was partially motivated by recent work [3] on spectral triples for more general Cantor sets. Our considerations closely resembled standard examples of geometric spectral triples that use the usual differentiation for the definition of the operator \mathcal{D} .

The geometric coarse-grained approximation of \mathbb{Z}_p , which we called the p -adic tree, is a weighted rooted tree $\{V, E\}$ associated to \mathbb{Z}_p via Michon's correspondence [3]. The set of vertices V of the p -adic tree consist of balls in \mathbb{Z}_p , with \mathbb{Z}_p itself being the root of the tree. There is an edge between two vertices v and v' if $v' \subset v$ and v' has the biggest diameter smaller than the diameter of v .

Now consider the Hilbert space H consisting of weighted ℓ^2 functions living on the vertices of the p -adic tree:

$$H = \{f : V \rightarrow \mathbb{C} : \sum_{v \in V} |f(v)|^2 w(v) < \infty\}. \quad (1.1)$$

Here the discrete-valued weight function $w : V \rightarrow \mathbb{R}_{\geq 0}$ is defined by: $w(v)$ = the volume of the ball v with respect to the additive Haar measure $d_p x$. For the space of p -adic integers \mathbb{Z}_p , the volume of a ball is equal to its diameter. It is useful to view H as the subspace of $L^2(\mathbb{Z}_p, d_p x)$ consisting of locally constant functions on \mathbb{Z}_p .

Then we introduced an unbounded operator D on H defined on its maximal domain $\{f \in H : Df \in H\}$ by

$$Df(v) = \frac{1}{\omega(v)} \left(f(v) - \frac{1}{(\deg v - 1)} \sum_{\substack{v' \in V \\ v' \sqsubset v}} f(v') \right), \quad (1.2)$$

where $\deg v$ is the degree of the vertex v and $v' \sqsubset v$ means that there is an edge between v and v' . One can think of D as a natural discrete derivative for (complex valued) functions

Date: March 11, 2015.

on \mathbb{Z}_p . This forward tree derivative was then used to construct the Dirac type operator \mathcal{D} , necessary for the spectral triple.

It was verified in [6] that the operator D is invertible with compact inverse, implying that D^*D has compact resolvent. Consequently, the spectrum of D^*D is discrete with only possible accumulation point at infinity.

In this paper our main interest is to find the spectrum of the operator D^*D . By re-parametrizing the vertices of the p -adic tree using the set of parameters $V \cong \mathcal{G}_p \times \mathbb{Z}_{\geq 0}$, where $\mathcal{G}_p = \mathbb{Q}_p/\mathbb{Z}_p$ is the Prüfer p -group, we can decompose H into invariant subspaces: $H = \bigoplus_{g \in \mathcal{G}_p} H_g$, where $H_g \cong \ell^2(\mathbb{Z}_{\geq 0})$. This allows the decomposition of the operators D and D^* into a direct sum of much simpler operators $D = \bigoplus_{g \in \mathcal{G}_p} D_g$, and $D^* = \bigoplus_{g \in \mathcal{G}_p} D_g^*$, where D_g is the restriction of D to H_g . Identifying the Prüfer group with the set of numbers $\{g = \frac{r}{p^m} : 0 \leq r < p^m, p \nmid r\}$, the operators D_g, D_g^* for $g = \frac{r}{p^m}$ can be written as $D_g = p^m D_0$ and $D_g^* = p^m D_0^*$ where D_0 is the operator on $\ell^2(\mathbb{Z}_{\geq 0})$ given by $D_0 f_n = p^n(f_n - f_{n+1})$. Consequently, $D^*D = \bigoplus_{g \in \mathcal{G}_p} p^{2m} D_0^* D_0$ and the problem of finding the spectrum of D^*D is reduced to finding the spectrum of $D_0^* D_0$.

It will be verified in this paper that the eigenvalues of $D_0^* D_0$ are the roots of the q -Bessel function ${}_1\phi_1\left(\begin{smallmatrix} 0 \\ q \end{smallmatrix}; q, \lambda\right)$ with $q = p^{-2}$. In [1] the authors give analytic bounds for these roots and discuss their asymptotic behavior. Therefore, we have a good understanding of the spectrum of D^*D . In particular, using the results of [1], we were able to obtain several results on analytic structure and analytic continuation of the zeta function of D^*D .

Part of the motivation for studying the spectrum of D^*D is that it may have some relevance for developing the structure of p -adic quantum mechanics. The operator D^*D , a natural analog of the laplacian, can be taken as an alternative starting point for the theory of p -adic Schrödinger operators, see [7].

The content of this paper is organized as follows. In section 2 we give a brief introduction to p -adic harmonic analysis and then describe the p -adic tree associated to the ring \mathbb{Z}_p . In the next section we analyze the forward derivative D on the p -adic tree and its adjoint D^* in H . We also describe the re-parametrization of the p -adic tree that leads to a decomposition of the operator D^*D into a direct sum of simpler operators $D_g^* D_g$. Section 4 discusses the calculation of the spectrum of $D_0^* D_0$ along with an “elliptic regularity” theorem that shows that the corresponding eigenfunctions are of a special form. In the last section we discuss some spectral properties of the operator D^*D and the analytic continuation of the zeta function associated to it.

2. DEFINITIONS AND NOTATION

2.1. Fourier Analysis in \mathbb{Z}_p . We start this section by briefly recalling some of the basic results and notation we introduced in [6] regarding harmonic analysis in the space \mathbb{Z}_p . For further reference and more details on this subject we refer to [5],[6] and [7].

The characters on \mathbb{Q}_p , the set of all p -adic numbers, are given by maps $\chi_a : \mathbb{Q}_p \rightarrow \mathbb{C}$ defined by $\chi_a(x) = e^{2\pi i \{ax\}}$ where $a \in \mathbb{Q}_p$, and $\{ax\}$ in the exponent is the fractional part of the p -adic number ax . Two characters $\chi_a(x)$ and $\chi_b(x)$ are equal on \mathbb{Z}_p if and only if $a - b \in \mathbb{Z}_p$. Consequently we see that the dual groups of \mathbb{Q}_p and \mathbb{Z}_p (denoted $\widehat{\mathbb{Q}_p}, \widehat{\mathbb{Z}_p}$) are $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$, $\widehat{\mathbb{Z}_p} = \mathbb{Q}_p/\mathbb{Z}_p$. The dual group of \mathbb{Z}_p , called the Prüfer group \mathcal{G}_p , can also be

identified with a group of roots of unity given by

$$\widehat{\mathbb{Z}_p} \cong \{e^{2\pi i \frac{k}{p^n}} : n \in \mathbb{Z}_{\geq 0}, p \nmid k \in \mathbb{Z}\}. \quad (2.1)$$

We let $\mathcal{E}(\mathbb{Z}_p)$ be the space of locally constant functions (test functions), i.e. the set of functions $\phi : \mathbb{Z}_p \rightarrow \mathbb{C}$ such that for every $x \in \mathbb{Z}_p$ there is a neighborhood U_x of x on which ϕ is constant. The space of linear functionals on $\mathcal{E}(\mathbb{Z}_p)$ (distributions on \mathbb{Z}_p) equipped with the weak*-topology is denoted by $\mathcal{E}^*(\mathbb{Z}_p)$.

If $d_p x$ denotes the Haar measure on $(\mathbb{Z}_p, +)$ normalized so that $\int_{\mathbb{Z}_p} d_p x = 1$, then we define the Fourier transform of a test function $\phi \in \mathcal{E}(\mathbb{Z}_p)$ as the function $\widehat{\phi}$ on $\widehat{\mathbb{Z}_p}$ given by

$$\widehat{\phi}([a]) = \int_{\mathbb{Z}_p} \phi(x) \overline{\chi_a(x)} d_p x.$$

For a locally constant function, only a finite number of Fourier coefficients will be nonzero. Thus, the Fourier transform gives an isomorphism between $\mathcal{E}(\mathbb{Z}_p)$ and $\mathcal{E}(\widehat{\mathbb{Z}_p})$, where the latter in our case is the space of all those functions on $\widehat{\mathbb{Z}_p}$ that are zero almost everywhere. The inverse Fourier transform is given by

$$\phi(x) = \sum_{[a] \in \widehat{\mathbb{Z}_p}} \widehat{\phi}([a]) \chi_a(x).$$

For a distribution $T \in \mathcal{E}^*(\mathbb{Z}_p)$ the Fourier transform is the function \widehat{T} on $\widehat{\mathbb{Z}_p}$ defined by $\widehat{T}([a]) = T(\overline{\chi_a(x)})$. Once again, the distributional Fourier transform gives an isomorphism between $\mathcal{E}^*(\mathbb{Z}_p)$ and $\mathcal{E}^*(\widehat{\mathbb{Z}_p})$. The inverse Fourier transform of a distribution is given by

$$T(\chi_a(x)) = \sum_{[a] \in \widehat{\mathbb{Z}_p}} \widehat{T}([a]) \chi_a(x).$$

2.2. The p -adic tree. We recall the construction of the weighted rooted tree $\{V, E\}$ associated to the Cantor metric space (\mathbb{Z}_p, ρ_p) , the space of p -adic integers equipped with the usual p -adic metric ρ_p , via Michon's correspondence [3], [6]. The symbols V and E above are used to denote the set of vertices and the edges of the tree, and we call this tree the p -adic tree.

The vertices of the p -adic tree are the balls in \mathbb{Z}_p . Since \mathbb{Z}_p is a totally disconnected space, the range of ρ_p is countable and consists of numbers of the form p^{-n} , $n \in \mathbb{Z}$ and zero. Therefore, if we let V_n be the set of balls of diameter p^{-n} then the set of vertices V has the natural decomposition $V = \bigcup_{n=0}^{\infty} V_n$. The set of edges E has the decomposition $E = \bigcup_{n=0}^{\infty} E_n$ where an edge $e = (v, v')$ between two vertices v, v' belongs to E_n if $v \in V_n$, $v' \in V_{n+1}$ and $v' \subset v$.

The compactness of the space \mathbb{Z}_p implies that the number of balls of diameter p^{-n} (hence the number of vertices) for fixed n , and the degree of each vertex are finite. Now we observe the following fact:

Proposition 2.1. *Every ball of radius p^{-n} contains a unique integer k such that $0 \leq k < p^n$.*

A proof of this proposition can be found in [6]. From this observation we see that there is a one-to-one correspondence between the set of integers $0 \leq k < p^n$ and the set V_n of balls of diameter p^{-n} . Therefore, the set of vertices has the natural parametrization:

$$V \cong S := \{(n, k) : n = 0, 1, 2, \dots, 0 \leq k < p^n\}. \quad (2.2)$$

Two vertices (n, k) and $(n+1, k')$ are connected by an edge if and only if $k' - k$ is divisible by p^n . Thus, a given vertex (n, k) will be connected (via edges) to exactly p vertices in V_{n+1} . Also, we introduce a weight function $w : V \rightarrow \mathbb{R}^+$ by $w(v) = \text{volume}(v)$ with respect to the Haar measure $d_p x$. If $v \in V_n$ then $w(v) = p^{-n}$.

3. THE OPERATOR D

3.1. A Forward Derivative on the p -adic tree. Due to the decomposition $V = \bigcup_{n=0}^{\infty} V_n$ of the set of vertices, any complex valued function f on V can be written as a sequence $\{f_n\}$ of complex valued functions on V_n . Let $\mathcal{E}^*(V)$ denote the space of all complex valued functions living on the vertices of the p -adic tree. By the discussion in the previous section we can identify each $V_n \cong \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$ with the finite additive group $\mathbb{Z}/p^n\mathbb{Z}$. Consequently, we can introduce the Fourier transform of a function $f \in \mathcal{E}^*(V)$ to be the discrete Fourier transform on each V_n given by

$$\widehat{f}_n(l) = \frac{1}{p^n} \sum_{k=0}^{p^n-1} f_n(k) e^{-2\pi i \frac{kl}{p^n}}, \quad 0 \leq l < p^n.$$

Because the characters of $\mathbb{Z}/p^n\mathbb{Z}$ satisfy the orthogonality condition:

$$\sum_{0 \leq s < p^j} e^{\frac{-2\pi i ks}{p^j}} = \begin{cases} 0 & \text{if } p^j \nmid k \\ p^j & \text{if } p^j \mid k, \end{cases} \quad (3.1)$$

we obtain the following Fourier inversion formula:

$$f_n(k) = \sum_{0 \leq l < p^n} \widehat{f}_n(l) e^{\frac{2\pi i kl}{p^n}}. \quad (3.2)$$

We also remark that the p -adic tree can be thought to be self dual, $\widehat{V} \cong V$, due to the fact that each $\mathbb{Z}/p^n\mathbb{Z}$ (hence V_n) is self dual. Thus, the Fourier transform on the p -adic tree is an isomorphism between the space $\mathcal{E}^*(V)$ of functions on the vertices of the p -adic tree and the space $\mathcal{E}^*(\widehat{V})$ of functions on the vertices of the dual tree. Additionally, via the Parseval's identity, the Fourier transform gives an isomorphism between the Hilbert space $H = \ell^2(V, w)$, of (1.1), and $\widehat{H} := \ell^2(\widehat{V})$, where the latter Hilbert space has no weight in the inner product.

Notice that the decomposition $V = \bigcup_{n=0}^{\infty} V_n$ induces a Hilbert space decomposition $H = \bigcup_{n=0}^{\infty} \ell^2(V_n, p^{-n})$. Using the parametrization (2.2) introduced in the previous section, we can write the action of the operator D of the formula (1.2) on the components f_n of f as:

$$(Df)_n(k) = p^n \left(f_n(k) - \frac{1}{p} \sum_{0 \leq j < p} f_{n+1}(k + jp^n) \right).$$

We note here that the choice of the domain for D , as well as all other unbounded operators below, is the maximal domain, i.e $\{f \in H : Df \in H\}$.

Using the Fourier transform of f_n and orthogonality of the characters (3.1) we write the following equivalent formula for D :

$$Df_n(k) = p^n \sum_{0 \leq l < p} \left(\widehat{f}_n(l) - \widehat{f}_{n+1}(pl) \right) e^{\frac{2\pi i k l}{p^n}}.$$

Hence, in Fourier transform, the operator D becomes \widehat{D} given by:

$$\widehat{D}\widehat{f}_n(l) = p^n (\widehat{f}_n(l) - \widehat{f}_{n+1}(pl)), \quad (3.3)$$

which is an unbounded operator on \widehat{H} . Notice that D and \widehat{D} are unitarily equivalent, but it is easier to work with the latter.

The adjoint D^* of D is given by

$$D^*g_n(k) = p^n \left[g_n(k) - \frac{1}{p} g_{n-1}(k \bmod p^{n-1}) \right], \quad (3.4)$$

assuming $g_{-1}(0) = 0$.

Later we will need the following formula for the adjoint \widehat{D}^* of \widehat{D} :

$$\widehat{D}^*\widehat{g}_n(l) = \begin{cases} p^n \widehat{g}_n(l) & \text{if } p \nmid l \\ p^n \left(\widehat{g}_n(l) - \frac{1}{p} \widehat{g}_{n-1}\left(\frac{l}{p}\right) \right) & \text{otherwise.} \end{cases} \quad (3.5)$$

It was verified in [6] that D and D^* are invertible with compact inverses.

3.2. Invariant Subspaces of H . The key observation that allows us to find the spectrum of D^*D is that we can decompose the Hilbert space H into invariant subspaces by means of a different parametrization of the p -adic tree.

The original parametrization (2.2) of the set of vertices of p -adic tree was done by using the set

$$S := \{(n, k) : n = 0, 1, 2, \dots, 0 \leq k < p^n\}.$$

Given a pair (n, k) in S notice that we can write $k = rp^l$ with $p \nmid r$ and $l \in \{0, 1, \dots, n-1\}$, by factoring out the highest power of p that divides k . Such a representation of k will be uniquely determined by r and l . If we associate n with $\frac{k}{p^n} = \frac{r}{p^{n-l}} = \frac{r}{p^m}$ where $m = n - l$, which is a unique representation of n in terms of r and m , then we have the correspondence $(n, k) \mapsto \left(\frac{r}{p^m}, l\right)$.

Conversely, given a pair $\left(\frac{r}{p^m}, l\right)$ where $0 \leq r < p^m$, $p \nmid r$ and $l \in \{0, 1, 2, \dots\}$ we can make the unique association $\left(\frac{r}{p^m}, l\right) \mapsto (m + l, rp^l)$. Thus, if

$$S' := \left\{ \left(\frac{r}{p^m}, l \right) : 0 \leq r < p^m, p \nmid r, l = 0, 1, 2, \dots \right\}$$

then we have the one-to-one correspondence between the sets S and S' given by $(n, k) \leftrightarrow \left(\frac{r}{p^m}, l\right)$.

In fact, the set of numbers $\left\{g = \frac{r}{p^m} : 0 \leq r < p^m, p \nmid r\right\}$ is isomorphic to the Prüfer group \mathcal{G}_p defined in (2.1). Therefore, $V \cong \widehat{V} \cong \mathcal{G}_p \times \mathbb{Z}_{\geq 0}$. Consequently we obtain the following new decomposition of the Hilbert space \widehat{H} :

$$\widehat{H} = \ell^2(S) \cong \ell^2(S') = \bigoplus_{\frac{r}{p^m} \in \mathcal{G}_p} \ell^2(\mathbb{Z}_{\geq 0}) =: \bigoplus_{g \in \mathcal{G}_p} \widehat{H}_g$$

where $\widehat{H}_g = \ell^2(\mathbb{Z}_{\geq 0})$.

We will now look at the operators \widehat{D} and \widehat{D}^* in the new coordinates. Using formula (3.3) we compute:

$$\widehat{D}\widehat{f}\left(\frac{r}{p^m}, l\right) = p^{m+l} \left(\widehat{f}\left(\frac{r}{p^m}, l\right) - \widehat{f}\left(\frac{r}{p^m}, l+1\right) \right).$$

Equation (3.5) yields:

$$\widehat{D}^*\widehat{f}\left(\frac{r}{p^m}, l\right) = \begin{cases} p^{m+l}\widehat{f}\left(\frac{r}{p^m}, 0\right) & \text{if } l = 0 \\ p^{m+l} \left(\widehat{f}\left(\frac{r}{p^m}, l\right) - \frac{1}{p}\widehat{f}\left(\frac{r}{p^m}, l-1\right) \right) & \text{otherwise.} \end{cases}$$

If we assume that $\widehat{f}\left(\frac{r}{p^m}, -1\right) = 0$ for any r, m then we can rewrite the formula for \widehat{D}^* as:

$$\widehat{D}^*\widehat{f}\left(\frac{r}{p^m}, l\right) = p^{m+l} \left(\widehat{f}\left(\frac{r}{p^m}, l\right) - \frac{1}{p}\widehat{f}\left(\frac{r}{p^m}, l-1\right) \right). \quad (3.6)$$

Notice that, in the new coordinates, the operators \widehat{D} and \widehat{D}^* affect only the second coordinate l and consequently each H_g is an invariant subspace. Thus, by letting $\widehat{D}_g := \widehat{D}|_{H_g}$ and $\widehat{D}_g^* := \widehat{D}^*|_{H_g}$ of \widehat{D}_g , we have the following decompositions of the operators \widehat{D} and \widehat{D}^* :

$$\widehat{D} = \bigoplus_{g \in \mathcal{G}_p} \widehat{D}_g \quad \text{and} \quad \widehat{D}^* = \bigoplus_{g \in \mathcal{G}_p} \widehat{D}_g^*. \quad (3.7)$$

Let \widehat{D}_0 be the operator on $\ell^2(\mathbb{Z}_{\geq 0})$ given by $\widehat{D}_0 f(l) = p^l (f(l) - f(l+1))$. It will be more convenient to switch to subscript notation and write:

$$(\widehat{D}_0 f)_n = p^n (f_n - f_{n+1}). \quad (3.8)$$

The adjoint of \widehat{D}_0 is given by

$$(\widehat{D}_0^* g)_n = p^n \left(g_n - \frac{1}{p} g_{n-1} \right). \quad (3.9)$$

From formula (3.6) we see that if $g = \frac{r}{p^m}$ then

$$\widehat{D}_g = p^m \widehat{D}_0 \quad \text{and} \quad \widehat{D}_g^* = p^m \widehat{D}_0^*.$$

Consequently, $\widehat{D}^* \widehat{D}$ has the decomposition;

$$\widehat{D}^* \widehat{D} = \bigoplus_{g \in \mathcal{G}_p} \widehat{D}_g^* \widehat{D}_g = \bigoplus_{g \in \mathcal{G}_p} p^{2m} (\widehat{D}_0^* \widehat{D}_0). \quad (3.10)$$

Thus, the key step in finding the spectrum of D^*D is to compute the spectrum of the operator $D_0^*D_0$ on $\ell^2(\mathbb{Z}_{\geq 0})$. We devote the next section to a description of this spectrum.

4. SPECTRUM OF $D_0^*D_0$

The fact that D^{-1} is compact implies that the operators D^*D and $D_0^*D_0$ have compact resolvent. Consequently, the spectrum of the unbounded operator $D_0^*D_0$ consists of eigenvalues diverging to infinity.

Using formulas (3.8) and (3.9) we obtain the following system of equations for $\widehat{D_0^*D_0}$.

$$\begin{aligned} (\widehat{D_0^*D_0}f)_0 &= f_0 - f_1 \\ (\widehat{D_0^*D_0}f)_n &= p^n \left((\widehat{D_0}f)_n - \frac{1}{p}(\widehat{D_0}f)_{n-1} \right) \\ &= p^{2n-2}[-p^2f_{n+1} + (1+p^2)f_n - f_{n-1}] \text{ for any } n \geq 1. \end{aligned}$$

We remark at this point that we could equivalently study the spectrum of $\widehat{D_0}\widehat{D_0^*}$; however the equations for the latter operator are not any simpler than the formulas for $\widehat{D_0^*D_0}$. Obviously, with the absence of kernels, the eigenvalue equations for both operators yield the same eigenvalues.

The problem is now to solve the following eigenvalue equations for $\widehat{D_0^*D_0}$:

$$\begin{aligned} p^{2n-2}[-p^2f_{n+1} + (1+p^2)f_n - f_{n-1}] &= \lambda f_n; \text{ for } n \geq 1 \\ f_0 - f_1 &= \lambda f_0, \end{aligned} \tag{4.1}$$

with $f_n \in \ell^2(\mathbb{Z}_{\geq 0})$.

The key step in solving the system of equations (4.1) is the following result which asserts that all eigenvectors of $\widehat{D_0^*D_0}$ take the special exponential sum form $f_n = \sum_{k=0}^{\infty} c(k)p^{-nk}$ with rapidly decaying coefficients $c(k)$. This result is a form of elliptic regularity of the operator $\widehat{D_0^*D_0}$.

Theorem 4.1. *Let $\{f_n(\lambda)\}$ be an eigenvector of $\widehat{D_0^*D_0}$ with eigenvalue λ . Then the following statements are true.*

- (1) *The sequence $\{f_n(\lambda)\}_{n=0}^{\infty}$ belongs to $\ell^1(\mathbb{Z}_{\geq 0})$.*
- (2) *The eigenvector $f_n(\lambda)$ can be uniquely expressed in the form*

$$f_n(\lambda) = \sum_{k=1}^{\infty} c(2k)p^{-2nk}, \tag{4.2}$$

where the coefficients $c(2k)$ decay exponentially in k .

- (3) *The coefficients $c(2k)$ satisfy the equations*

$$c(2) = \left(\frac{\lambda}{1-p^{-2}} \right) \sum_{k=0}^{\infty} f_k,$$

and, for $k \geq 2$,

$$c(2k) = \left(\frac{-\lambda}{1-p^{-2}} \right)^{k-1} \frac{c(2)p^{k(k-1)}(p^2-1)^{k-2}}{(p^4-1)^2(p^6-1)^2 \dots (p^{2k-2}-1)^2(p^{2k}-1)}. \tag{4.3}$$

(4) If the remainder $r_n(2N)$ is defined by the formula:

$$f_n(\lambda) = c(2)p^{-2n} + c(4)p^{-4n} + c(6)p^{-6n} + \dots + c(2N-2)p^{-(2N-2)n} + r_n(2N),$$

then $\{r_n(2N)\}_{n=0}^\infty \rightarrow 0$ as $N \rightarrow \infty$ in the ℓ^1 norm.

Proof. The main idea of the proof is to rewrite the equations (4.1) in an integral equation form and then use it iteratively to produce the solution. To this end we regroup the terms in the first equation of system (4.1) above to obtain:

$$(f_n - f_{n-1}) - p^2(f_{n+1} - f_n) = \lambda p^{2-2n} f_n.$$

Using the notation $\Delta f_n := f_{n+1} - f_n$, we can then rewrite the system of equations (4.1) as follows.

$$\begin{aligned} \Delta f_n &= p^{-2}(\Delta f_{n-1} - \lambda p^{2-2n} f_n) \quad \text{for } n \geq 1 \\ \Delta f_0 &= f_1 - f_0 = -\lambda f_0. \end{aligned} \tag{4.4}$$

Iteratively, with the help of equations (4.4), we obtain the following formula for Δf_n :

$$\Delta f_n = -p^{-2n} \lambda (f_0 + f_1 + \dots + f_n), \quad n \geq 0. \tag{4.5}$$

Equation (4.5) is a one-step linear difference equation, so it has one-parameter family of solutions. However, since we are looking for the solution in the Hilbert space we need to choose one that vanishes at infinity. This leads to the following formula for f_n :

$$f_n = \sum_{l=n}^{\infty} \lambda p^{-2l} \sum_{k=0}^l f_k.$$

Interchanging the summation indices of the above formula we obtain:

$$\begin{aligned} f_n &= \sum_{k=0}^n \sum_{l=n}^{\infty} \lambda p^{-2l} f_k + \sum_{k=n+1}^{\infty} \sum_{l=k}^{\infty} \lambda p^{-2l} f_k \\ &= \frac{\lambda}{(1-p^{-2})} \left[p^{-2n} \sum_{k=0}^n f_k + \sum_{k=n+1}^{\infty} p^{-2k} f_k \right]. \end{aligned} \tag{4.6}$$

Thus we can estimate:

$$\sum_{n=0}^{\infty} |f_n| \leq \frac{\lambda}{(1-p^{-2})} \left[\sum_{n=0}^{\infty} p^{-2n} \sum_{k=0}^n |f_k| + \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} p^{-2k} |f_k| \right].$$

By interchanging the summation indices in the first sum above and evaluating the sum over n we obtain:

$$\sum_{n=0}^{\infty} p^{-2n} \sum_{k=0}^n |f_k| = \sum_{k=0}^{\infty} |f_k| \left(\frac{p^{-2k}}{1-p^{-2}} \right).$$

Using Cauchy-Schwartz inequality and the fact that $f \in \ell^2(\mathbb{N})$ we conclude that this sum is bounded:

$$\sum_{k=0}^{\infty} |f_k| \left(\frac{p^{-2k}}{1-p^{-2}} \right) \leq \left(\frac{1}{1-p^{-2}} \right) \sqrt{\frac{1}{1-p^{-4}}} \|f\|_2 < \infty.$$

Notice that for the second sum we have:

$$\sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} p^{-2k} |f_k| = \sum_{n=0}^{\infty} p^{-2n} \sum_{l=1}^{\infty} p^{-2l} |f_{n+l}|.$$

Once again using Cauchy-Schwartz inequality we see that the second sum is finite:

$$\sum_{n=0}^{\infty} p^{-2n} \sum_{l=1}^{\infty} p^{-2l} |f_{n+l}| \leq \sqrt{\frac{p^{-4}}{1-p^{-4}}} \left(\frac{1}{1-p^{-2}} \right) \|f\|_2 < \infty.$$

This verifies that $\{f_n\} \in \ell^1(\mathbb{N})$.

To prove the second part of Theorem 4.1, we observe that equation (4.6) gives;

$$f_n = \frac{\lambda p^{-2n}}{(1-p^{-2})} \left[\sum_{k=0}^{\infty} f_k - \sum_{k=n+1}^{\infty} f_k + \sum_{l=1}^{\infty} p^{-2l} f_{n+l} \right].$$

Rearranging the terms on the right hand side of the equation to isolate the coefficient of p^{-2n} we get:

$$f_n = \left(\frac{\lambda}{1-p^{-2}} \sum_{k=0}^{\infty} f_k \right) p^{-2n} - \lambda p^{-2n} \sum_{l=1}^{\infty} \left(\frac{1-p^{-2l}}{1-p^{-2}} \right) f_{n+l}, \quad (4.7)$$

from which we extract the coefficient

$$c(2) := \left(\frac{\lambda}{1-p^{-2}} \right) \sum_{k=0}^{\infty} f_k.$$

Notice that $c(2)$ is well defined due to part (1).

Recursively applying this formula for f_n on the right hand side of equation (4.7) we obtain:

$$f_n = c(2) p^{-2n} - \lambda p^{-2n} \sum_{l=1}^{\infty} \left(\frac{1-p^{-2l}}{1-p^{-2}} \right) \left(c(2) p^{-2n-2l} - \lambda p^{-2n-2l} \sum_{k=1}^{\infty} \left(\frac{1-p^{-2k}}{1-p^{-2}} \right) f_{n+l+k} \right).$$

Once again we rearrange the terms to extract the coefficient $c(4)$ of p^{-4n} .

$$\begin{aligned} f_n &= c(2) p^{-2n} + \left(-\lambda c(2) \sum_{l=1}^{\infty} \left(\frac{p^{-2l} - p^{-4l}}{1-p^{-2}} \right) \right) p^{-4n} \\ &\quad + \frac{\lambda^2}{p^{4n}} \sum_{l=1}^{\infty} \left(\frac{p^{-2l} - p^{-4l}}{1-p^{-2}} \right) \sum_{k=1}^{\infty} \left(\frac{1-p^{-2k}}{1-p^{-2}} \right) f_{n+l+k}. \end{aligned}$$

This gives:

$$c(4) = \frac{-\lambda c(2)}{(1-p^{-2})} \cdot \frac{p^2}{(p^4-1)}.$$

By repeatedly applying this process we can obtain an expansion of f_n in powers of p^{-2n} , provided the remainder $r_n(2N)$ goes to zero as $N \rightarrow \infty$. We prove a stronger ℓ^1 estimate on $r_n(2N)$ below, implying the pointwise convergence needed for the existence of the expansion of f_n .

Using induction we readily establish that the coefficients $c(2k)$ of this expansion are in general given by the formula:

$$c(2k) = \left(\frac{-\lambda}{1-p^{-2}} \right)^{k-1} \frac{c(2)p^{k(k-1)}(p^2-1)^{k-2}}{(p^4-1)^2(p^6-1)^2 \dots (p^{2k-2}-1)^2(p^{2k}-1)} . \quad (4.8)$$

Next we estimate the growth of the coefficients $c(2k)$. Simplifying the formula for $c(2k)$ we obtain:

$$\begin{aligned} c(2k) &= \frac{c(2)\lambda^{k-1}}{p^{(k-2)(k-1)}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^4}\right)^2\left(1-\frac{1}{p^6}\right)^2 \dots \left(1-\frac{1}{p^{2k-2}}\right)^2\left(1-\frac{1}{p^{2k}}\right)} \\ &= \frac{\lambda^k\left(1-\frac{1}{p^{2k}}\right) \sum_{m=0}^{\infty} f_m}{p^{(k-2)(k-1)}\left(1-\frac{1}{p^2}\right)^2\left(1-\frac{1}{p^4}\right)^2\left(1-\frac{1}{p^6}\right)^2 \dots \left(1-\frac{1}{p^{2k-2}}\right)^2\left(1-\frac{1}{p^{2k}}\right)^2} . \end{aligned}$$

Since $\prod_{i=1}^k (1-p^{-2i})^2 \geq \prod_{i=1}^{\infty} (1-p^{-2i})^2$ and the infinite product is a finite nonzero number, we obtain the following estimate for the coefficients $c(2k)$:

$$|c(2k)| \leq \frac{|\lambda|^k \|f\|_1}{p^{(k-2)(k-1)} \prod_{i=1}^{\infty} (1-p^{-2i})^2} ,$$

which shows that they decay exponentially. This establishes both the second and the third part of the theorem.

Finally we estimate the remainder term $r_n(2N)$. Using induction it is easily established that the remainder $r_n(2N)$ satisfies the following summation formula:

$$\begin{aligned} r_n(2N) &= \frac{(-\lambda)^N}{p^{2nN}} \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \dots \sum_{l_N=1}^{\infty} \frac{(p^{-(2N-2)l_1} - p^{-2Nl_1})}{1-p^{-2}} \frac{(p^{-(2N-4)l_2} - p^{-(2N-2)l_2})}{1-p^{-2}} \\ &\quad \dots \frac{(1-p^{-2l_N})}{1-p^{-2}} f_{n+l_1+l_2+\dots+l_N} . \end{aligned}$$

Estimating the ℓ^1 norm we see that:

$$\begin{aligned} \sum_{n=1}^{\infty} |r_n(2N)| &\leq \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \dots \sum_{l_N=1}^{\infty} \frac{|\lambda|^N}{p^{2nN}(1-p^{-2})^N} (p^{-(2N-2)l_1} - p^{-2Nl_1})(p^{-(2N-4)l_2} - p^{-(2N-2)l_2}) \\ &\quad \dots (1-p^{-2l_N}) |f_{n+l_1+l_2+\dots+l_N}| . \end{aligned} \quad (4.9)$$

Notice that the term $\sum_{l_N=1}^{\infty} (1-p^{-2l_N}) |f_{n+l_1+l_2+\dots+l_N}|$ can be estimated as follows:

$$\begin{aligned} \sum_{l_N=1}^{\infty} (1-p^{-2l_N}) |f_{n+l_1+l_2+\dots+l_N}| &\leq \sup_{l_N \geq 1} (1-p^{-2l_N}) \sum_{l_N=1}^{\infty} |f_{n+l_1+l_2+\dots+l_N}| \\ &\leq \|f\|_1 \end{aligned}$$

where the last line can be justified by changing the summation index in the previous line appropriately. Moreover, we can explicitly calculate each sum that appears in formula (4.9). For example:

$$\begin{aligned} \sum_{n=1}^{\infty} p^{-2Nn} &= \frac{p^{-2N}}{1 - p^{-2N}}, \\ \sum_{l_1=1}^{\infty} (p^{-(2N-2)l_1} - p^{-2Nl_1}) &= \frac{(1 - p^{-2})p^{-(2N-2)}}{(1 - p^{-(2N-2)})(1 - p^{-2N})}, \\ \sum_{l_2=1}^{\infty} (p^{-(2N-4)l_2} - p^{-(2N-2)l_2}) &= \frac{(1 - p^{-2})p^{-(2N-4)}}{(1 - p^{-(2N-4)})(1 - p^{-(2N-2)})}, \end{aligned}$$

and so on. Substituting all these values into the formula (4.9) we get the following estimate:

$$\sum_{n=1}^{\infty} |r_n(2N)| \leq \frac{|\lambda|^N}{(1 - p^{-2})^N} \cdot \frac{p^{-2N}}{1 - p^{-2N}} \cdot \frac{(1 - p^{-2})p^{-(2N-2)}}{(1 - p^{-(2N-2)})(1 - p^{-2N})} \cdots \frac{(1 - p^{-2})p^{-2}}{(1 - p^{-4})(1 - p^{-2})}.$$

Simplifying this expression we obtain:

$$\sum_{n=1}^{\infty} |r_n(2N)| \leq \frac{|\lambda|^N}{p^{N(N+1)} \prod_{k=1}^N (1 - p^{-2k})^2} \leq \frac{|\lambda|^N}{p^{N(N+1)} \prod_{k=1}^{\infty} (1 - p^{-2k})^2}.$$

Since $\prod_{k=1}^N (1 - p^{-2k})^2 < \infty$ we see that $\sum_{n=1}^{\infty} |r_n(2N)| \rightarrow 0$ as $N \rightarrow \infty$.

Finally we will prove uniqueness of the expansion of $f_n(\lambda)$. Consider the analytic function $f(z) = \sum_{k=1}^{\infty} c(2k)z^k$. From the above estimate of the coefficients $c(2k)$ we see that the radius of convergence R of the power series for f is given by:

$$\frac{1}{R} = \limsup_{k \rightarrow \infty} \sqrt[k]{|c(2k)|} \leq |\lambda| \limsup_{k \rightarrow \infty} \sqrt[k]{\frac{\|f\|_1}{\prod_{i=1}^{\infty} (1 - p^{-2i})^2}} \cdot \frac{1}{p^{k-3+2/k}}.$$

Therefore, $R = \infty$ and the function $f(z)$ is entire. Therefore, in particular:

$$f(p^{-2n}) = \sum_{k=1}^{\infty} c(2k)p^{-2nk} = f_n(\lambda),$$

and so the coefficients $c(2k)$ are uniquely determined by $f_n(\lambda)$, because an analytic function is completely determined by its values on a convergent sequence of points, [2]. □

Remark: The collection of ℓ^2 functions with a power series representation of the form (4.2) is fairly restrictive which is clear from the fact that $\lim_{n \rightarrow \infty} p^{2n} f(n) = c(2)$. It can be easily shown that the set of ℓ^2 functions with this power series representation is dense in the space of all ℓ^2 functions.

The difficult part already completed, we can now state our main theorem.

Theorem 4.2. *The spectrum of the operator $\widehat{D}_0^* \widehat{D}_0$ consists of simple eigenvalues $\{\lambda_n\}$ which are the roots of the q -hypergeometric function $\lambda \mapsto {}_1\phi_1\left(\begin{smallmatrix} 0 \\ q \end{smallmatrix}; q, \lambda\right)$, with $q = \frac{1}{p^2}$.*

Proof. Substituting $f_n = \sum_{k=1}^{\infty} c(2k)p^{-2nk}$ and formula (4.8) into the initial condition of system (4.1) and dividing throughout by $c(2)$ we obtain the following:

$$\frac{1}{p^2} + \lambda - 1 + \sum_{k=2}^{\infty} \frac{(p^{-2k} + \lambda - 1) \lambda^{k-1} p^{2k-2}}{\prod_{j=2}^k (1 - p^{2j}) \left(1 - \frac{1}{p^{2j-2}}\right)} = 0. \quad (4.10)$$

The infinite sum on the left hand side of the above equation, call it S_1 , can be simplified by first breaking it up into two terms, extracting some terms and then recombining as follows:

$$\begin{aligned} S_1 &= \sum_{k=2}^{\infty} \frac{(p^{-2k} - 1) \lambda^{k-1} p^{2k-2}}{\prod_{j=2}^k (1 - p^{2j}) \left(1 - \frac{1}{p^{2j-2}}\right)} + \sum_{k=3}^{\infty} \frac{\lambda^{k-1} p^{2k-4}}{\prod_{j=2}^{k-1} (1 - p^{2j}) \left(1 - \frac{1}{p^{2j-2}}\right)} \\ &= \frac{\left(\frac{1}{p^4} - 1\right) \lambda p^2}{(1 - p^4) \left(1 - \frac{1}{p^2}\right)} + \sum_{k=3}^{\infty} \frac{p^{2k-4} \lambda^{k-1} \left[p^2(p^{-2k} - 1) + (1 - p^{2k}) \left(1 - \frac{1}{p^{2k-2}}\right)\right]}{\prod_{j=2}^k (1 - p^{2j}) \left(1 - \frac{1}{p^{2j-2}}\right)}. \end{aligned}$$

Using the substitution $q = \frac{1}{p^2}$ equation (4.10) can be written as

$$(q - 1) + \frac{\lambda}{(1 - q)} + \sum_{k=3}^{\infty} \frac{q^{2-k} \lambda^{k-1}}{\prod_{j=2}^{k-1} (1 - q^{-j}) \prod_{j=2}^k (1 - q^{j-1})} = 0.$$

Notice that at $k = 2$ the expression

$$\frac{q^{2-k} \lambda^{k-1}}{\prod_{j=2}^{k-1} (1 - q^{-j}) \prod_{j=2}^k (1 - q^{j-1})}$$

yields the value $\frac{\lambda}{(1-q)}$. Thus the above equation is in fact equal to:

$$(q - 1) + \sum_{k=2}^{\infty} \frac{q^{2-k} \lambda^{k-1}}{\prod_{j=2}^{k-1} (1 - q^{-j}) \prod_{j=2}^k (1 - q^{j-1})} = 0.$$

Now we rearrange the terms in the infinite sum in order to compare it with the hypergeometric function ${}_1\phi_1\left(\begin{smallmatrix} 0 \\ q \end{smallmatrix}; q, \lambda\right)$.

$$\begin{aligned} (q - 1) + \sum_{k=2}^{\infty} \frac{q^{2-k} \lambda^{k-1}}{\prod_{j=2}^{k-1} (1 - q^{-j}) \prod_{j=2}^k (1 - q^{j-1})} &= (q - 1) + \sum_{k=2}^{\infty} \frac{(-1)^k \lambda^{k-1} (1 - q) q^{\frac{(k-2)(k-1)}{2}}}{\prod_{j=1}^{k-1} (1 - q^j)^2} \\ &= (q - 1) - \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k (1 - q) q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^k (1 - q^j)^2} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^k (1 - q^j)^2}. \end{aligned}$$

By using the notation

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$$

and the above computation, we can rewrite the eigenvalue equation as

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k q^{\frac{k(k-1)}{2}}}{(q; q)_k^2} = 0.$$

The function ${}_1\phi_1$ of four variables a_0, b_1, q, z is defined as

$${}_1\phi_1 \left(\begin{matrix} a_0 \\ b_1 \end{matrix} ; q^2, z \right) = \sum_{n=0}^{\infty} \frac{(a_0; q^2)_n}{(q^2; q^2)_n (b_1; q^2)_n} (-1)^n q^{2\binom{n}{2}} z^n.$$

Thus, if λ is an eigenvalue, we get:

$${}_1\phi_1 \left(\begin{matrix} 0 \\ q \end{matrix} ; q, \lambda \right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k q^{\frac{k(k-1)}{2}}}{(q; q)_k^2} = 0,$$

showing that the eigenvalues of the operator $\widehat{D}_0^* \widehat{D}_0$ are the roots of the above q -hypergeometric function. Conversely, the above calculation shows that given a root λ of $\lambda \mapsto {}_1\phi_1 \left(\begin{matrix} 0 \\ q \end{matrix} ; q, \lambda \right)$ the formula (4.2) with arbitrary $c(2)$ and other coefficients $c(2k)$ given by (4.3) gives, up to a constant, the unique eigenvector of $\widehat{D}_0^* \widehat{D}_0$ corresponding to eigenvalue λ . By the analysis in [6] the whole spectrum of $\widehat{D}_0^* \widehat{D}_0$ consists of such eigenvalues. \square

5. SPECTRAL PROPERTIES

5.1. Spectrum of D^*D . Computation of the spectrum of D^*D is based on decomposition (3.10) and the analysis of the spectrum of $\widehat{D}_0^* \widehat{D}_0$ in the previous section.

Theorem 5.1. *Let $\{\lambda_n\}$ be the eigenvalues of the operator $\widehat{D}_0^* \widehat{D}_0$ and let $\widehat{D}_g^* \widehat{D}_g$ be as in formula (3.7). Then,*

- (1) *The spectrum of $\widehat{D}_g^* \widehat{D}_g$ consists of simple eigenvalues $\{p^{2m} \lambda_n\}$ i.e., $\sigma(\widehat{D}_g^* \widehat{D}_g) = \bigcup_n \{p^{2m} \lambda_n\}$.*
- (2) *$\sigma(D^*D) = \sigma(\widehat{D}^* \widehat{D}) = \bigcup_{m,n} \{p^{2m} \lambda_n\}$. Moreover, each eigenvalue of $\widehat{D}^* \widehat{D}$ occurs with multiplicity $p^m(1 - \frac{1}{p})$.*

Proof. The above results follow directly from the decomposition (3.10). Since the number of different values of r less than p^m that are relatively prime to p is equal to $p^m - p^{m-1}$, each eigenvalue of $\widehat{D}^* \widehat{D}$ in H has multiplicity $p^m(1 - \frac{1}{p})$. \square

Corollary 5.2. *The operator $(D^*D)^{-1}$ is a s -th Schatten class operator for all $s \geq 1$.*

Proof. From the decomposition (3.10) we see that:

$$(D^*D)^{-s} = \bigoplus_{\frac{r}{p^m} \in \mathcal{G}_p} p^{-2ms} (D_0^* D_0)^{-s},$$

from which we compute the following trace:

$$\begin{aligned} \mathrm{Tr}(D^*D)^{-s} &= \sum_{\substack{r \\ p^m \nmid r} \in \mathcal{G}_p} p^{-2ms} \mathrm{Tr}(D_0^*D_0)^{-s} \\ &= \sum_{m=0}^{\infty} \sum_{\substack{0 \leq r < p^m \\ p \nmid r}} p^{-2ms} \mathrm{Tr}(D_0^*D_0)^{-s}. \end{aligned}$$

Since the number of nonnegative r 's less than p^m and relatively prime to p is equal to the Euler number of p^m , we can compute the sum over m provided that $s > \frac{1}{2}$:

$$\begin{aligned} \mathrm{Tr}(D^*D)^{-s} &= \sum_{m=0}^{\infty} (p^m - p^{m-1}) p^{-2ms} \mathrm{Tr}(D_0^*D_0)^{-s} \\ &= \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{1-2s}}\right) \mathrm{Tr}(D_0^*D_0)^{-s}. \end{aligned} \tag{5.1}$$

From [1] we have that $\lambda_n \leq p^n$, so we can estimate the trace $\mathrm{Tr}(D_0^*D_0)^{-s} = \sum_{n=0}^{\infty} (\lambda_n)^{-s}$ as follows, provided $s > 0$:

$$\mathrm{Tr}(D_0^*D_0)^{-s} = \sum_{n=0}^{\infty} (\lambda_n)^{-s} \leq \sum_{n=0}^{\infty} p^{-ns} = \frac{1}{1 - p^{-s}}.$$

Summing up this information we see that,

$$\mathrm{Tr}(D^*D)^{-s} \leq \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{1-2s}}\right) \left(\frac{1}{1 - p^{-s}}\right)$$

whenever $s > \frac{1}{2}$. Thus for any $s \geq 1$ the s -th Schatten norm of $(D^*D)^{-1}$ is finite. \square

5.2. Analytic continuation of the zeta functions. Using formula (5.1) we can express the zeta function associated with the operator D^*D , denoted $\zeta_D(s)$, in terms of $\zeta_{D_0}(s)$, the zeta function associated with the operator $D_0^*D_0$:

$$\zeta_D(s) = \left(1 - \frac{1}{p}\right) \left(\frac{1}{1 - p^{1-2s}}\right) \zeta_{D_0}(s). \tag{5.2}$$

We now consider the analytic continuation of $\zeta_{D_0}(s)$.

Theorem 5.3. $\zeta_{D_0}(s)$ is holomorphic for $\mathrm{Re} s > 0$ and can be analytically continued to a meromorphic function for $\mathrm{Re} s > -2$.

Proof. To show that $\zeta_{D_0}(s)$ is holomorphic in the region $\mathrm{Re} s > 0$ we estimate:

$$|\zeta_{D_0}(s)| \leq \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n^{\mathrm{Re} s + i \mathrm{Im} s}} \right| = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\mathrm{Re} s}},$$

since $\lambda_n^{-i \mathrm{Im} s}$ is unimodular. From [1] we know that the eigenvalues λ_n of $D_0^*D_0$ satisfy the following upper and lower bounds:

$$p^n \left(1 - \frac{p^{-2n}}{1 - p^{-2n}} \right) < \lambda_n < p^n. \quad (5.3)$$

Thus we get:

$$|\zeta_{D_0}(s)| \leq \sum_{n=1}^{\infty} \frac{1}{\left(p^n \left(1 - \frac{p^{-2n}}{1 - p^{-2n}} \right) \right)^{\operatorname{Re} s}}.$$

We have an elementary inequality:

$$\frac{p^{-2n}}{1 - p^{-2n}} = 1 - \frac{1}{p^{2n} - 1} \geq \frac{p^2 - 2}{p^2 - 1},$$

which holds since the left-hand side is an increasing function of n , while the right-hand side is its value at $n = 1$. Therefore, we get:

$$|\zeta_{D_0}(s)| \leq \left(\frac{p^2 - 1}{p^2 - 2} \right)^{\operatorname{Re} s} \sum_{n=1}^{\infty} \frac{1}{p^{n \operatorname{Re} s}},$$

which is convergent for $\operatorname{Re} s > 0$. Consequently, $\zeta_{D_0}(s)$ is holomorphic in $\operatorname{Re} s > 0$.

We now show that $\zeta_{D_0}(s)$ can be analytically continued to $\operatorname{Re} s > -2$. Since λ_n behaves like p^n , the analytic continuation of $\zeta_{D_0}(s)$ will be achieved by a perturbative argument from the meromorphic function obtained from the zeta function by replacing λ_n with p^n . First we write:

$$p^{-ns} - \lambda_n^{-s} = e^{-sn \ln p} - e^{-s \ln(\lambda_n)} = \int_{-\ln(\lambda_n)}^{-n \ln p} \frac{d}{dt} e^{ts} dt = s \int_{-\ln(\lambda_n)}^{-n \ln p} e^{ts} dt.$$

Thus, we obtain:

$$|p^{-ns} - \lambda_n^{-s}| \leq |s| \int_{-\ln(\lambda_n)}^{-n \ln p} |e^{ts}| dt = |s| \int_{-\ln(\lambda_n)}^{-n \ln p} e^{t \operatorname{Re} s} dt.$$

In this integral we can estimate the integrand by its maximum on the interval of integration $[-\ln(\lambda_n), -n \ln p]$ to arrive at the following estimate:

$$|p^{-ns} - \lambda_n^{-s}| \leq \begin{cases} |s|(n \ln p - \ln(\lambda_n)) e^{-n \ln p \operatorname{Re} s} & \text{if } \operatorname{Re} s \leq 0 \\ |s|(n \ln p - \ln(\lambda_n)) e^{-\ln(\lambda_n) \operatorname{Re} s} & \text{if } \operatorname{Re} s > 0. \end{cases}$$

Inequality (5.3) implies that:

$$\ln \left(\frac{p^n}{\lambda_n} \right) < -\ln \left(1 - \frac{p^{-2n}}{1 - p^{-2n}} \right) = \frac{p^{-2n}}{1 - p^{-2n}} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{1}{p^{2n} - 1} \right)^k.$$

Since $1 - p^{-2n} > \frac{1}{2}$ for $n \geq 1$, we can estimate the above as:

$$\ln \left(\frac{p^n}{\lambda_n} \right) < 2p^{-2n} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{1}{p^2 - 1} \right)^k = -2p^{-2n} \ln \left(1 - \frac{1}{p^2 - 1} \right).$$

Consequently, if $\operatorname{Re} s \leq 0$, we have:

$$|p^{-ns} - \lambda_n^{-s}| \leq -2|s| \ln \left(1 - \frac{1}{p^2 - 1} \right) p^{-2n} p^{-n\operatorname{Re} s}.$$

This lets us estimate the difference of the series as follows:

$$\left| \sum_{n=1}^{\infty} (p^{-ns} - \lambda_n^{-s}) \right| \leq -2 \ln \left(1 - \frac{1}{p^2 - 1} \right) |s| \sum_{n=1}^{\infty} p^{-n(2+\operatorname{Re} s)}.$$

The series $\sum_{n=1}^{\infty} p^{-n(2+\operatorname{Re} s)}$ is convergent for $\operatorname{Re} s > -2$ hence, by the Weierstrass M test, the series $\sum_{n=1}^{\infty} (p^{-ns} - \lambda_n^{-s})$ converges uniformly for $\operatorname{Re} s > -2$ and hence it is analytic for $\operatorname{Re} s > -2$.

Moreover, since

$$\sum_{n=1}^{\infty} p^{-ns} = \frac{p^{-s}}{1 - p^{-s}}$$

is meromorphic in the complex plane with poles at $s = \frac{2\pi ik}{\ln p}$, $k \in \mathbb{Z}$, we obtain that the zeta function $\sum_{n=1}^{\infty} \lambda_n^{-s}$ is meromorphic for $\operatorname{Re} s > -2$ with the above mentioned poles. \square

Corollary 5.4. $\zeta_D(s)$ is meromorphic for $\operatorname{Re} s > -2$ with poles at $s = \frac{2\pi ik}{\ln p}$, and $s = \frac{1}{2} \left(1 - \frac{2\pi ik}{\ln p} \right)$, where $k \in \mathbb{Z}$.

Proof. The proof of this corollary follows from the theorem above and equation (5.2). \square

6. APPENDIX

In this section we record some basic properties and identities satisfied by the q -hypergeometric function ${}_1\phi_1$ we encountered in section 4. More on q -hypergeometric functions can be found in [4]. We start with the general definition of these type of functions:

$${}_{r+1}\phi_s \left(\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_0; q)_n (a_1; q)_n \dots (a_r; q)_n}{(q; q)_n (b_1; q)_n \dots (b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r} z^n$$

where $b_j \neq q^{-n}$ for any j, n .

Here we used the notation $(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$. We remark that $(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j)$. When $s > r$ the above series converges for all z while it converges for $|z| < 1$ when $s = r$.

We are interested in the special case where $r = 0, s = 1$ and $a = 0, b = q$, which leads to the formula:

$${}_1\phi_1 \left(\begin{matrix} 0 \\ q \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{1}{(q; q)_n (q; q)_n} (-1)^n q^{\binom{n}{2}} z^n.$$

The function ${}_1\phi_1$ satisfies the Cauchy's sum:

$${}_1\phi_1 \left(\begin{matrix} a \\ b \end{matrix} ; q, b/a \right) = \frac{(b/a; q)_{\infty}}{(b; q)_{\infty}}.$$

In [1] the authors investigated the roots of the third Jackson q -Bessel function:

$$J_\nu^{(3)}(z; q) := z^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} ; q, qz^2 \right),$$

where $0 < q < 1$ and z is a complex parameter. It is known that this function has infinitely many zeros, each of multiplicity one, all of them real. When $\nu = 0$ we have that the third Jackson q -Bessel function equals the function ${}_1\phi_1$ which we used in this paper.

We record the following transformation property of ${}_1\phi_1$:

$${}_1\phi_1 \left(\begin{matrix} 0 \\ b \end{matrix} ; q, z \right) = \frac{(z; q)_\infty}{(b; q)_\infty} {}_1\phi_1 \left(\begin{matrix} 0 \\ z \end{matrix} ; q, b \right).$$

Starting with this transformation the authors in [1] deduce that if $q < (1 - q)^2$ then the positive roots $\omega_k(q)$, $k = 1, 2, 3, \dots$, of $J_0^{(3)}(z; q)$, arranged in the increasing order satisfy the following:

$$q^{-k/2 + \alpha_k(q)} < \omega_k(q) < q^{-k/2},$$

where

$$\alpha_k(q) = \frac{\log \left(1 - \frac{q^k}{1 - q^k} \right)}{\log q}.$$

In particular, this gives the asymptotic behavior $\omega_k \sim q^{-k/2}$ as $k \rightarrow \infty$. Additionally, those results give an upper and lower bound (5.3) for the roots of the specific ${}_1\phi_1$ function needed in this paper.

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